Chapter 8
Network Models

to accompany

Introduction to Mathematical Programming: Operations Research, Volume 1
4th edition, by Wayne L. Winston and Munirpallam Venkataramanan

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Description

Many important optimization problems can be analyzed by means of graphical or network representation. In this chapter the following network models will be discussed:

1. Shortest path problems
2. **Maximum flow problems**
3. CPM-PERT project scheduling models
4. **Minimum cost network flow problems**
5. **Minimum spanning tree problems**

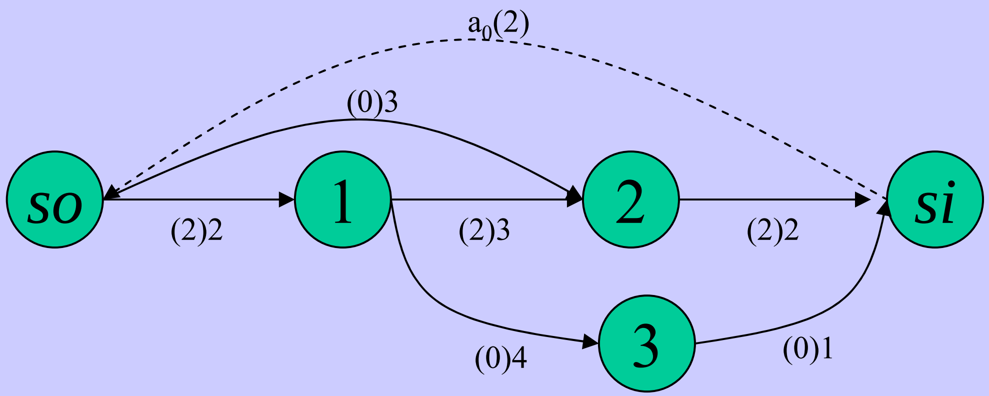
8.3 Maximum Flow Problems

Many situations can be modeled by a network in which the arcs may be thought of as having a capacity that limits the quantity of a product that may be shipped through the arc.

In these situations, it is often desired to transport the maximum amount of flow from a starting point (called the source) to a terminal point (called the sink). Such problems are called maximum flow problems.

An example for maximum flow problem

Sunco Oil wants to ship the maximum possible amount of oil (per hour) via pipeline from node *so* to node *si* as shown in the figure below.



Arc	*Capacity*
(*so*,1)	2
(*so*,2)	3
(1,2)	3
(1,3)	4
(3,*si*)	1
(2,*si*)	2

The various arcs represent pipelines of different diameters. The maximum number of barrels of oil that can be pumped through each arc is shown in the table above (also called arc capacity).

For reasons that will become clear soon, an artificial arc called a_0 is added from the sink to the source. To formulate an LP about this problem first we should determine the decision variable.

X_{ij} = Millions of barrels of oil per hour that will pass through arc(i,j) of pipeline.

For a flow to be feasible it needs to be in the following range:

0 <= flow through each arc <= arc capacity

(arc capacity constraints)

And

Flow into node i = Flow out from node i

(balance flow constraints)

Let X_0 be the flow through the artificial arc, the conservation of flow implies that X_0 = total amount of oil entering the sink. Thus, Sunco's goal is to maximize X_0.

Max Z= X_0

S.t. $X_{so,1} <= 2$ *(Arc Capacity constraints)*

$X_{so,2} <= 3$

$X_{12} <= 3$

$X_{2,si} <= 2$

$X_{13} <= 4$

$X_{3,si} <= 1$

$X_0 = X_{so,1} + X_{so,2}$ *(Node so flow constraints)*

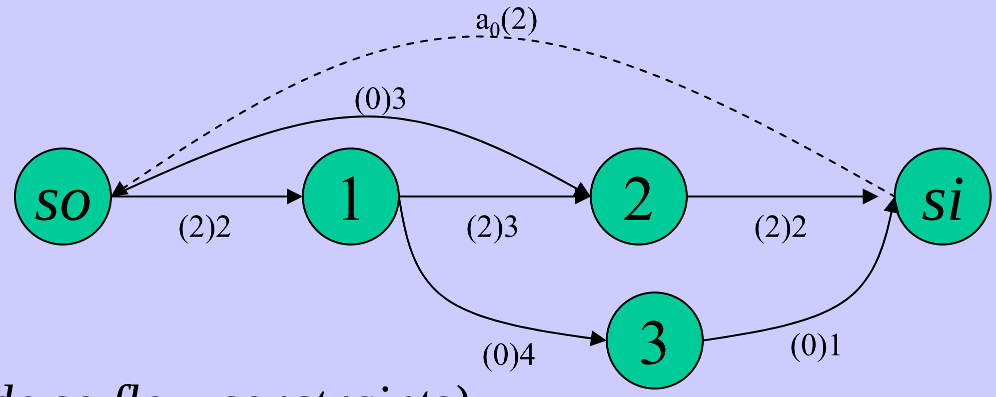
$X_{so,1} = X_{12} + X_{13}$ *(Node 1 flow constraints)*

$X_{so,2} + X_{12} = X_{2,si}$ *(Node 2 flow constraints)*

$X_{13} + X_{3,si}$ *(Node 3 flow constraints)*

$X_{3,si} + X_{2,si} = X_0$ *(Node si flow constraints)*

$X_{ij} >= 0$



One optimal solution to this LP is Z=3, $X_{so,1}=2$, $X_{13}=1$, $X_{12}=1$, $X_{so,2}=1$, $X_{3,si}=1$, $X_{2,si}=2$, $X_o=3$.

Note: The Ford-Fulkerson Algorithm is used for solving max flow problems.

8.5 Minimum Cost Network Flow Problems

The transportation, assignment, transshipment, shortest path, maximum flow, and CPM problems are all special cases of minimum cost network flow problems (MCNFP). Any MCNFP can be solved by a generalization of the transportation simplex called the network simplex.

To define MCNFP, let

X_{ij}= number of units of flow sent from node i to node j through arc(i,j)

b_i= net supply (outflow-inflow) at node i

c_{ij}= cost of transporting 1 unit of flow from node i to node j via arc(i,j)

L_{ij}= lower bound of flow through arc(i,j) (if there is no lower bound, let L_{ij}=0)

U_{ij}= upper bound of flow through arc(i,j) (if there is no upper bound, let U_{ij}=infinity)

Then the MCNFP can be modeled as follows:

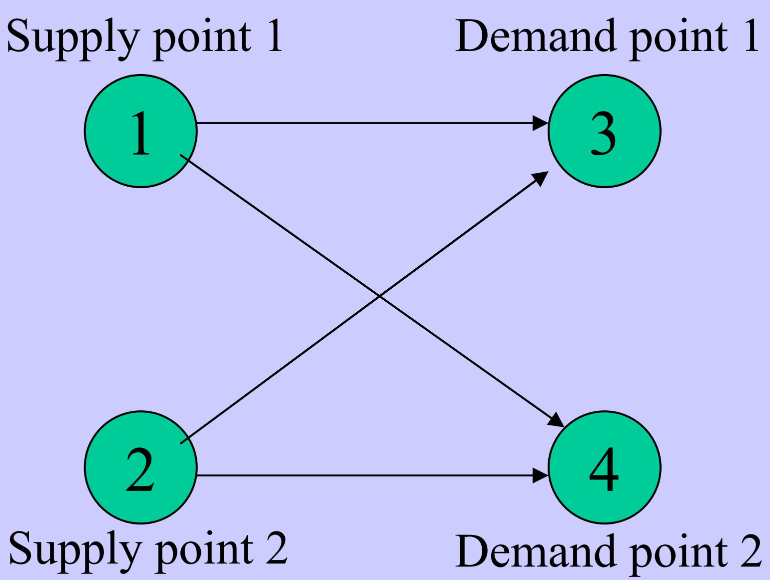
$$\text{Min} \sum_{allarcs} c_{ij} X_{ij}$$

$$\text{s.t.} \quad \sum_{j} X_{ij} - \sum_{k} X_{ki} = b_i \quad \text{(for each node i in the network)}$$

$$L_{ij} \leq X_{ij} \leq U_{ij} \quad \text{(for each arc in the network)}$$

Formulating a transportation problem as an MCNFP

Consider the transportation problem below:



	6 (Node 3)	3 (Node 4)	
	1	2	4 (Node 1)
	3	4	5 (Node 2)

MCNFP Representation of the problem:

$$\text{Min } Z = X_{13}+2X_{14}+3X_{23}+4X_{24}$$

X_{13}	X_{14}	X_{23}	X_{24}		rhs	Constraint
1	1	0	0	=	4	Node 1
0	0	1	1	=	5	Node 2
-1	0	-1	0	=	-6	Node 3
1	-1	0	-1	=	-3	Node 4

All Variables nonnegative

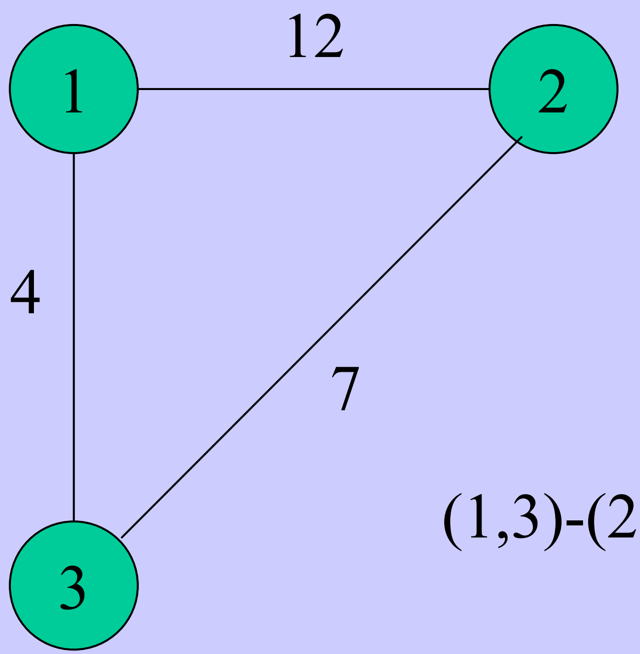
8.6 Minimum Spanning Tree Problems

Suppose that each arc (i,j) in a network has a length associated with it and that arc (i,j) represents a way of connecting node i to node j.

For example, if each node in a network represents a computer in a computer network, arc(i,j) might represent an underground cable that connects computer i to computer j.

In **many applications** (electricity, computers, ICs, etc), we want to determine the set of arcs in a network that connect all nodes such that the sum of the length of the arcs is minimized. Clearly, such a group of arcs contain no loop.

For a network with n nodes, a spanning tree is a group of n-1 arcs that connects all nodes of the network and contains no loops.



(1,2)-(2,3)-(3,1) is a loop

(1,3)-(2,3) is the minimum spanning tree

The following method (MST Algorithm) may be used to find a minimum spanning tree:

• Begin at any node i, and join node i to the node in the network (call it node j) that is closest to node i. The two nodes i and j now form a connected set of nodes C={i,j}, and arc (i,j) will be in the minimum spanning tree. The remaining nodes in the network (call them Ć) are referred to as the unconnected set of nodes.

The Minimum Spanning Tree (MST) Algorithm

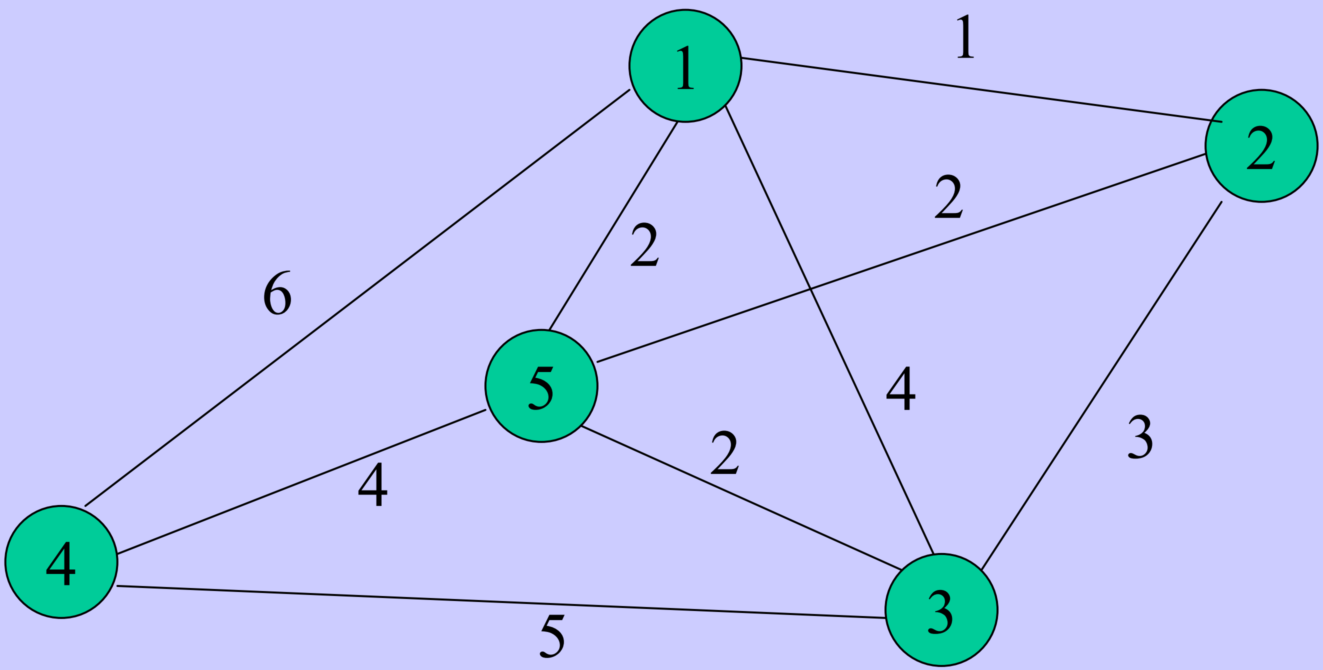
- Now choose a member of Ć (call it n) that is closest to some node in C. Let m represent the node in C that is closest to n. Then the arc (m,n) will be in the minimum spanning tree.

- Now update C and Ć. Since n is now connected to {i,j}, C now equals {i,j,n} and we must eliminate node n from Ć.

- Repeat this process until a minimum spanning tree is found.

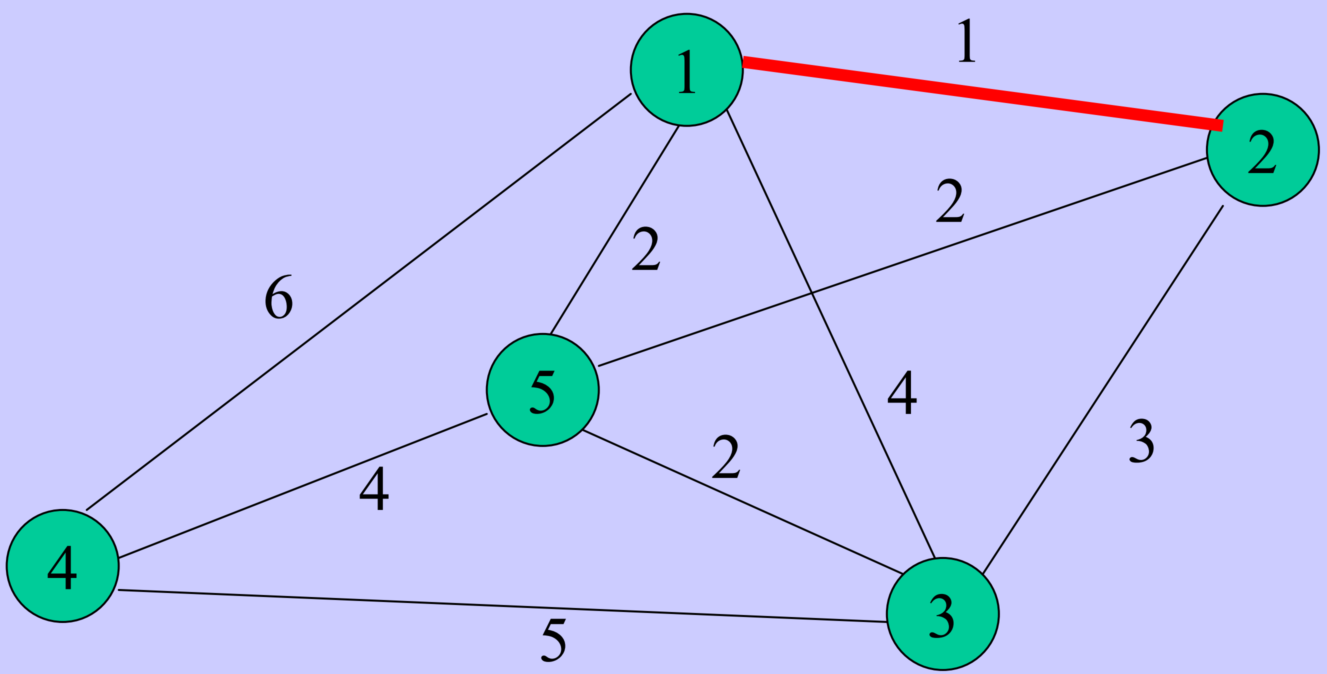
- Ties for closest node and arc to be included in the minimum spanning tree may be broken arbitrarily.

Example: The State University campus has five computers. The distances between computers are given in the figure below. What is the minimum length of cable required to interconnect the computers? Note that if two computers are not connected this is because of underground rock formations.

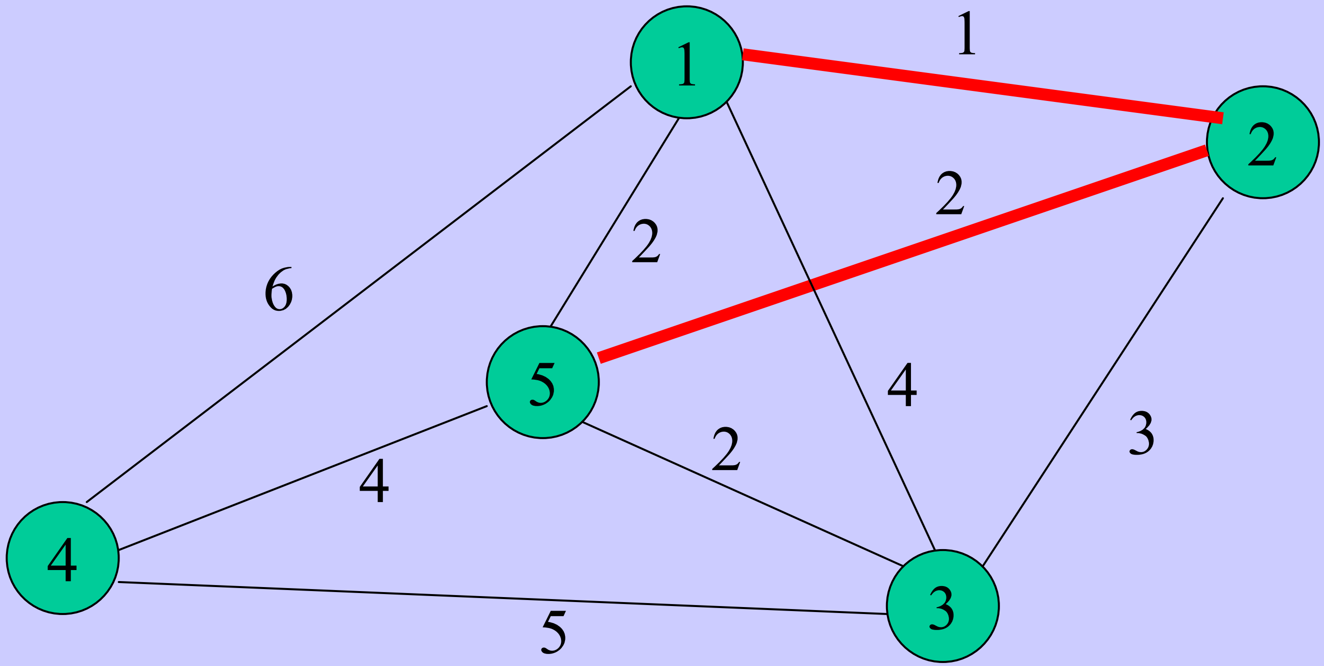


Solution: We want to find the minimum spanning tree.

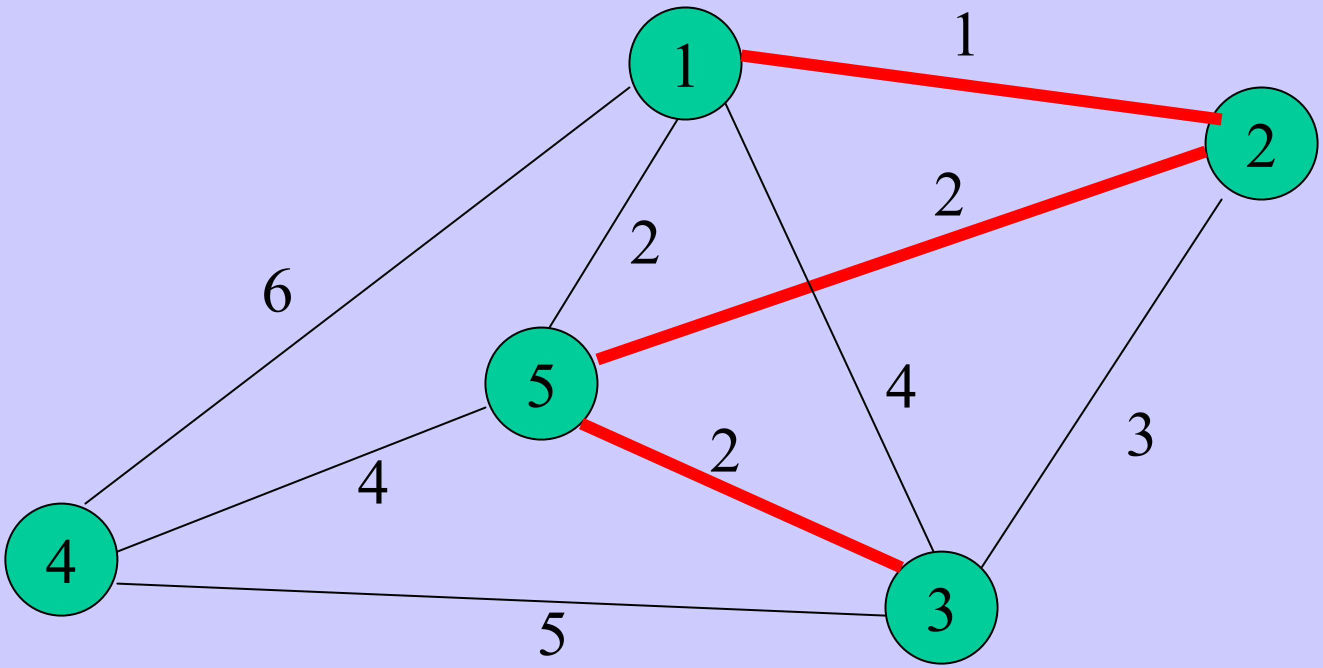
- Iteration 1: Following the MST algorithm discussed before, we arbitrarily choose node 1 to begin. The closest node is node 2. Now C={1,2}, Ć={3,4,5}, and arc (1,2) will be in the minimum spanning tree.



- Iteration 2: Node 5 is closest to C. since node 5 is two blocks from node 1 and node 2, we may include either arc(2,5) or arc(1,5) in the minimum spanning tree. We arbitrarily choose to include arc(2,5). Then C={1,2,5} and Ć={3,4}.



- Iteration 3: Since node 3 is two blocks from node 5, we may include arc(5,3) in the minimum spanning tree. Now C={1,2,5,3} and Ć={4}.



• Iteration 4: Node 5 is the closest node to node 4. Thus, we add arc (5,4) to the minimum spanning tree.

We now have a minimum spanning tree consisting of arcs (1,2), (2,5), (5,3), and (5,4). The length of the minimum spanning tree is 1+2+2+4=9 blocks.